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Math Paper III DR. ANJANI KUMAR SINGH.

Topic - Different ways of defining a group and their equivalence.

Theorem :- Define group in two different ways and show their equivalency.

Solution :- 1st way — Let G be a non-empty set, o be any binary operation then the system (G, o) is called group iff following laws hold.

(i) Closure law ; $\forall\ a, b \in G$ such that $aob \in G$

(ii) Associative law :- $\forall\ a, b, c \in G$, such that

$$(aob)oc = ao(boc)$$

(iii) Existence of identity : $\forall\ a \in G$, $aoe = eoa = a$

(iv) Existence of inverse : $\forall\ a \in G$, $aoa^{-1} = a^{-1}oa = e$

2nd way - A system (G, o) where o is binary operation is said to be group if following postulates holds:

(i) closure law : $\forall\ a, b \in G$ then $aob \in G$

(ii) Associative law : $\forall\ a, b, c \in G$ then $(aob)oc = ao(boc)$

(iii) Uniqueness of solution :- The equations $aox = b$ and $yoa = b$ have unique solution in G $\forall\ a, b \in G$.

Equivalence of two definitions : We observe that law (i) and (ii) of Ist definition and 2nd definition are simil

For a group (G, o) if $a, b \in G$ then there exist a uni element $x \in G$ such that $aox = b$ and for $y \in G$ such tha

$y \circ a = b$ i.e. law (iii) of 2nd definition.

Thus we arive in a position that the postulates of the second definition have been derived from the postulates of the first definition. We remain to derive the postulates of Ist definition assumption of 2nd definition.

From the (i) and (ii) law of 2nd def we get (i) and (ii) of Ist definition.

So we have to derive (iii) and (iv) laws of Ist definition from the laws of second definition.

From (iii) laws of 2nd definition there exists x_1, x such that

$$x_1 \circ a = a \text{ and } a \circ x_1 = a \qquad \text{——— (1)}$$

Also $\exists\ x_3, x_4$, $a \circ x_3 = b$ and $x_4 \circ a = b$ ——— (2)
where $b \in G$

We have

$$\begin{aligned} b \circ x_1 &= (x_4 \circ a) \circ x_1 && \text{by (2)} \\ &= x_4 \circ (a \circ x_1) && \text{by (ii)} \\ &= x_4 \circ a && \text{by (1)} \end{aligned}$$

$\Rightarrow b \circ x_1 = b$ (by (2))

Now taking x_2 for b as b is arbitrary $\Rightarrow x_2 \circ x_1 = x_2$ ——— (3)

We have by (2)

$$\begin{aligned} x_2 \circ b &= x_2 \circ (a \circ x_3) && \text{——— by (ii)} \\ &= (x_2 \circ a) \circ x_3 && \text{——— by (1)} \\ &= a \circ x_3 && \text{——— by (2)} \end{aligned}$$

$\Rightarrow x_2 \circ b = b$, Taking $x_1 = b$ as b is arbitrary

$\therefore x_2 \circ x_1 = x_1$ ——— (4)

By eq^n (3) and (4)

$$x_1 = x_2 = x_2 \circ x_1 = e \text{ (say)}$$

With regards to (III) this is unique

So by (i), $a \circ e = e \circ a = a$

∴ e is the unique identity element in G

i.e. identity law of first definition.

Lastly by (III) suppose c is the unique solution of

$a \circ x = e$ (where e is identity element)

$$\Rightarrow a \circ c = e \quad \text{——— (5)}$$

and if d is the unique solution of

$$x \circ a = e$$

$$\Rightarrow d \circ a = e \quad \text{——— (6)}$$

$$\Rightarrow (d \circ a) \circ c = e \circ c$$

$\Rightarrow d \circ (a \circ c) = c$ (By associativity and e is identity)

$\Rightarrow d \circ e = c$ by (5)

$$\Rightarrow d = c$$

So by (5) and (6)

$$a \circ c = c \circ a = e$$

∴ c is the inverse element of a. Thus every element of G has an inverse element in G.

So by supposition of 2nd definition, we find all postulates of first definition satisfied by G under the operation o.

∴ (G, o) is a group.

So the two definitions of [illegible]